

## RAMSEY-PRODUCT SUBSETS OF A GROUP

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ABSTRACT. We say that a subset  $S$  of an infinite group  $G$  is a Ramsey-product subset if, for any infinite subsets  $X, Y$  of  $G$ , there exist  $x \in X$  and  $y \in Y$  such that  $xy \in S$  and  $yx \in S$ . We show that the family  $\varphi$  of all Ramsey-product subsets of  $G$  is a filter and  $\varphi$  defines the subsemigroup  $\overline{G^*G^*}$  of the semigroup  $G^*$  of all free ultrafilters on  $G$ .

All groups under consideration are supposed to be infinite; a countable set means a countably infinite set.

We say that a subset  $S$  of a group  $G$  is

- a *Ramsey-product subset* if, for any infinite subsets  $X, Y$  of  $G$ , there exist  $x \in X$  and  $y \in Y$  such that  $xy \in S$  and  $yx \in S$ ;
- a *Ramsey-square subset* if, for every infinite subset  $X$  of  $G$  there exist distinct elements  $x, y \in X$  such that  $xy \in S$ ,  $yx \in S$ ;
- a *Ramsey-quotient subset* if, for every infinite subset  $X$  of  $G$ , there exist distinct elements  $x, y \in X$  such that  $x^{-1}y \in S$  and  $y^{-1}x \in S$ .

We show that above defined subsets of  $G$  arise naturally in context of the Stone-Čech compactification  $\beta G$  of the group  $G$  endowed with the discrete topology. We identify  $\beta G$  with the set of all ultrafilters on  $G$ . Then the family  $\{\bar{A} : A \subseteq G\}$ , where  $\bar{A} = \{p \in \beta G : A \in p\}$ , forms a base for the topology of  $\beta G$ . Given a filter  $\varphi$  on  $G$ , we denote  $\bar{\varphi} = \bigcap \{\bar{A} : A \in \varphi\}$ , so  $\varphi$  defines the closed subset  $\bar{\varphi}$  of  $\beta G$ , and every non-empty closed subset of  $\beta G$  can be defined in this way.

We use the standard extension [3, Section 4.1] of the multiplication on  $G$  to the semigroup multiplication on  $\beta G$ . Given two ultrafilters  $p, q \in \beta G$ , we choose  $P \in p$  and, for each  $x \in P$ , pick  $Q_x \in q$ . Then  $\bigcup_{x \in P} xQ_x \in pq$  and the family of these subsets forms a base of the product  $pq$ . We note that the set  $G^*$  of all free ultrafilters of  $G$  is a closed subsemigroup of  $\beta G$ , and the closure  $\overline{G^*G^*}$  of  $G^*G^*$  is a subsemigroup of  $G^*$ . An ultrafilter  $p \in G^* \setminus \overline{G^*G^*}$  is called *strongly prime*.

By [5, Propositions 3, 4], the family  $\psi$  of all Ramsey-quotient subsets of a group  $G$  is a filter and  $\bar{\psi}$  is the smallest closed subset of  $G^*$  containing all ultrafilters of the form  $q^{-1}q$ ,  $q \in G^*$ , where  $q^{-1} = \{Q^{-1} : Q \in q\}$ . Analogously, the closure of  $\{pp : p \in G^*\}$  is defined by the filter of all Ramsey-square subsets of  $G$ . In both cases, we used the classical Ramsey theorem [2, p. 16].

In this note, the key technical part plays the following version of Ramsey theorem (Theorem 6 from [2, p.98]).

*Let  $\chi$  be a finite coloring  $\chi : \omega \times \omega \rightarrow [r]$ . Then there exists an infinite set  $A = \{a_i\}_{i < \omega}$  and colors  $c_L, c_G, c_E$  (not necessarily distinct) such that  $\chi(a_i, a_j) = c_L$  if  $i < j$ ,  $\chi(a_i, a_j) = c_G$  if  $i > j$  and  $\chi(a_i, a_j) = c_E$  if  $i = j$ .*

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**Lemma 1.** *If  $S$  is a Ramsey-product subsets of a group  $G$  then, for any countable subsets  $X, Y$  of  $G$ , there exist disjoint countable subsets  $X' \subseteq X$  and  $Y' \subseteq Y$  such that  $X'Y' \subseteq S$ ,  $Y'X' \subseteq S$ .*

*Proof.* Passing to subsets of  $X$  and  $Y$ , we may suppose that  $X \cap Y = \emptyset$ . We enumerate  $X = \{x_n : n < \omega\}$ ,  $Y = \{y_n : n < \omega\}$  and define a  $\{0, 1\}$ -coloring  $\chi$  of  $\omega \times \omega$  by the rule:  $\chi((n, m)) = \chi((m, n)) = 1$  if  $m > n$ ,  $x_n y_m \in S$ ,  $y_m x_n \in S$ , and  $\chi((m, n)) = 0$  otherwise. By Theorem 6 from [2, p.98], there exists a countable subset  $A$  of  $\omega$  such that  $\chi$  is monochrome on all  $(n, m)$ , such that  $n \in A$ ,  $m \in A$ ,  $n \neq m$ . We partition  $A$  into two infinite subsets  $A = B \cup C$  and denote  $X' = \{x_n : n \in B\}$ ,  $Y' = \{y_n : n \in C\}$ . Since  $S$  is a Ramsey-product set, there are  $x' \in X'$ ,  $y' \in Y'$  such that  $x'y' \in S$ ,  $y'x' \in S$ . By the choice of  $A$ , we have  $xy \in S$ ,  $yx \in S$  for all  $x \in X'$ ,  $y \in Y'$ .  $\square$

**Lemma 2.** *For a free ultrafilter  $r$  on a group  $G$ ,  $r \in \overline{G^*G^*}$  if and only if, for every subset  $R \in r$ , there exist two countable subsets  $\{x_n : n < \omega\}$  and  $\{y_n : n < \omega\}$  of  $G$  such that  $\{x_n y_m : n \leq m < \omega\} \subseteq R$ .*

*Proof.* We assume that  $r \in \overline{G^*G^*}$ , take an arbitrary  $R \in r$  and choose  $p, q \in G^*$  such that  $R \in pq$ . We choose  $P \in p$  and  $Q_x$ ,  $x \in P$  such that  $\bigcup_{x \in P} xQ_x \subseteq R$ . We take an arbitrary countable subset  $\{x_n : n < \omega\}$  of  $P$  and choose an injective sequence  $(y_n)_{n \in \omega}$  in  $G$  such that  $y_n \in Q_{x_0} \cap \dots \cap Q_{x_n}$ ,  $n < \omega$ . Then  $\{x_n y_m : n \leq m < \omega\} \subseteq R$ .

On the other hand, let  $R \in r$  and  $\{x_n : n \in \omega\}$ ,  $\{y_n : n \in \omega\}$  are chosen so that  $\{x_n y_m : n \leq m < \omega\} \subseteq R$ . We take an arbitrary free ultrafilters  $p, q$  such that  $\{x_n : n \in \omega\} \in p$ ,  $\{y_n : n \in \omega\} \in q$ . Then  $\{x_n y_m : n \leq m < \omega\} \in pq$ , so  $R \in pq$  and  $r \in \overline{G^*G^*}$ .  $\square$

We recall [1] that a subset  $A$  of a group  $G$  is *sparse* if, for every infinite subset  $X$  of  $G$ , there exists a non-empty finite subset  $F \subset X$  such that  $\bigcap_{x \in F} xA$  is finite. By [1, Theorem 9], an ultrafilter  $r \in G^*$  is strongly prime if and only if there exists a sparse subset  $A$  of  $G$  such that  $A \in r$ . For sparse subsets see also [4], [6].

**Theorem.** *For every group  $G$ , the following statements hold:*

- (i) *the family  $\varphi$  of all Ramsey-product subsets of  $G$  is a filter;*
- (ii)  *$\overline{\varphi} = \overline{G^*G^*}$ ;*
- (iii)  *$A \in \varphi$  if and only if  $G \setminus A$  is sparse.*

*Proof.* (i) We take two Ramsey-product subsets  $S$  and  $T$  of  $G$  and show that  $S \cap T \in \varphi$ . Let  $X, Y$  be infinite subsets of  $G$ . By Lemma 1, there exist disjoint countable subsets  $X' \subseteq X$  and  $Y' \subseteq Y$  such that  $xy \in S$ ,  $yx \in S$  for all  $x \in X'$ ,  $y \in Y'$ . Since  $T$  is a Ramsey-product subsets, there exist  $x' \in X'$ ,  $y' \in Y'$  such that  $x'y' \in T$ ,  $y'x' \in T$ . Then  $x'y' \in S \cap T$ ,  $y'x' \in S \cap T$ , so  $S \cap T \in \varphi$ .

(ii) To show that  $\overline{G^*G^*} \subseteq \overline{\varphi}$ , we suppose the contrary, choose  $p \in \overline{G^*G^*} \setminus \overline{\varphi}$  and take  $P \in p$  such that  $G \setminus P \in \varphi$ . By Lemma 2, there are countable sets  $\{x_n : n < \omega\}$  and  $\{y_n : n < \omega\}$  such that  $\{x_n y_m : n \leq m < \omega\} \subseteq P$ . We use Lemma 1 to choose disjoint countable subsets  $X' \subseteq \{x_n : n < \omega\}$ ,  $Y' \subseteq \{y_n : n < \omega\}$  such that  $X'Y' \subseteq G \setminus P$ ,  $Y'X' \subseteq G \setminus P$ . To get a contradiction, we take an arbitrary  $x_n \in X'$  and pick  $y_m \in Y'$  such that  $m > n$ .

Now we take an arbitrary  $p \in \overline{\varphi}$  and prove that  $p \in \overline{G^*G^*}$ . Given any  $P \in p$ , by Lemma 1, there exist two disjoint countable subsets  $X, Y$  of  $G$  such that, for any countable subsets  $X' \subseteq X$  and  $Y' \subseteq Y$ , either  $X'Y' \cap P \neq \emptyset$  or  $Y'X' \cap P \neq \emptyset$ . We enumerate  $X = \{x_n : n < \omega\}$ ,  $Y = \{y_n : n < \omega\}$  and define a  $\{0, 1\}$ -coloring  $\chi$  of  $\omega \times \omega$  by the rule:  $\chi(n, m) = 1$  if and only if  $n < m$  and  $x_n y_m \in P$  or  $n > m$  and  $y_m x_n \in P$ . Applying Theorem 6 from [2, p. 98], we get an infinite subset  $W$  of  $\omega$  such that the restriction of  $\chi$  to  $W_1 = \{(n, m) : n < m, n \in W, m \in W\}$  is monochrome, and the restriction of  $\chi$  to  $W_2 = \{(n, m) : n > m, n \in W, m \in W\}$  is monochrome. By the choice of  $X$  and  $Y$ , either  $\chi|_{W_1} \equiv 1$  or  $\chi|_{W_2} \equiv 1$ . In the first case, we choose two injective sequences  $(x'_n)_{n < \omega}$  in  $X$  and  $(y'_n)_{n < \omega}$  in  $Y$  such that  $\{x'_n y'_m : n < m < \omega\} \subseteq P$ . In the second case, we choose two injective sequences  $(x'_n)_{n < \omega}$  in  $X$  and  $(y'_n)_{n < \omega}$  in  $Y$  such that  $\{y'_n x'_m : n < m < \omega\} \subseteq P$ . Applying Lemma 2, we conclude that  $p \in \overline{G^*G^*}$ .

(iii) We apply (ii) and Theorem 9 from [1].  $\square$

We recall that a subset  $S$  of a group  $G$  is *large (extralarge)* if there is a finite subset  $K$  of  $G$  such that  $G = KS$  ( $S \cap L$  is large for every large subset  $L$  of  $G$ ). By Theorem 4.1 from [4], the complement of sparse subset is extralarge. Applying (iii), we see that each Ramsey-product subset is extralarge.

By the statement (ii), every Ramsey-product subset of  $G$  is a member of every idempotent of the semigroup  $G^*$ , in particular, every idempotent from the minimal ideal of  $\beta G$ , so  $S$  is combinatorially very rich (see [3, Section 14]).

If  $G$  is an amenable group and  $A$  is a sparse subset of  $G$ , by Theorem 5.1 from [4],  $\mu(A) = 0$  for every left invariant Banach measure  $\mu$  on  $G$ . Applying (iii), we see that  $\mu(S) = 1$  for every Ramsey-product subset  $S$  of  $G$  and every left invariant Banach measure  $\mu$  on  $G$ .

We conclude the note with the following formal generalization. We fix two infinite subset  $X, Y$  of a group  $G$  and say that a subset  $S$  of  $G$  is a *Ramsey  $(X, Y)$ -product subset* if, for any infinite subsets  $X' \subseteq X$  and  $Y' \subseteq Y$ , there exist  $x \in X'$  and  $y \in Y'$  such that  $xy \in S$ ,  $yx \in S$ . Then the family  $\varphi_{X,Y}$  of all Ramsey  $(X, Y)$ -product subsets of  $G$  is a filter and  $\overline{\varphi}_{X,Y} = \overline{X^*Y^*} \cup \overline{Y^*X^*}$ .

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